

# Generalized Landau damping due to multi-plasmon resonances

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We study wave-particle interaction of Langmuir waves in a fully degenerate plasma using the Wigner-Moyal equation. As is well known, in the short wavelength regime the resonant velocity is shifted from the phase velocity due to the finite energy and momentum of individual plasmon quanta. In the present work we focus on the case when the resonant velocity lies outside the background distribution, i.e. when it is larger than the Fermi velocity. Going beyond the linearized theory we show that we can still have nonlinear wave-particle damping associated with multi-plasmon resonances. Sets of evolution equations are derived for the case of two-plasmon resonance and for the case of three-plasmon resonance. The damping rates of the Langmuir waves are deduced for both cases, and the implications of the results are discussed.

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The process of Landau damping has been of long-standing interest in plasma physics, where the case of Langmuir waves has been the prominent example. For the most part Landau damping has been described classically. When the initial amplitude of the Langmuir waves is increased, the constant damping rate from linear theory is changed to amplitude oscillations with the bounce frequency of trapped particles [1]. Various aspects of this have been studied in a classical context e.g. by Refs. [2–4], also including the evolution towards a BGK equilibrium [5] and effects induced when the wave mode is driven [6]. When quantum effects are included, new mechanisms enter the picture. This includes for example the effect of the electron spin on Landau damping [7], and nonlinear modifications of the Fermi surface [8].

In the present letter our starting point is the Wigner-Poisson system [9–13]. For this system quantum effects can be seen to a certain extent already in a "weak" quantum regime, when the Langmuir wavelength is much longer than the characteristic de Broglie wavelength. An example of such effects are bounce-like oscillations in the absence of trapped particles [12]. However, in the weak quantum regime the resonant particles still have a velocity close to the phase velocity of the wave, just as in classical theory. In the strong quantum regime, when the de Broglie wavelength is of the same order as the wavelength, the resonant particles fulfill  $\omega - kv_z \pm \hbar k^2/2m \simeq 0$  in linearized theory [10, 13]. Here  $\omega$  and  $\mathbf{k} = k\hat{\mathbf{z}}$  are the

wave frequency and wavevector,  $v_z$  is the electron velocity along  $\hat{\mathbf{z}}$ ,  $\hbar = 2\pi\hbar$  is Planck's constant, and  $m$  is the electron mass. For a fully degenerate plasma the maximum velocity of the background distribution is the Fermi velocity  $v_F$ . Thus linear Landau damping will take place if and only if the wavelength is short enough such that the resonance condition is fulfilled for some  $v_z < v_F$ . This holds provided  $k > k_{cr}$ , where the critical wavenumber  $k_{cr}$  is computed in Ref. [10]. However, the picture changes if we study the nonlinear evolution, allowing for multi-plasmon resonances. As will be demonstrated, resonant wave-particle interaction can take place if the resonance condition  $\omega - kv_z \pm n\hbar k^2/2m \simeq 0$  is fulfilled, where  $n$  is an integer. The case with  $n = 1$  corresponds to one-plasmon processes,  $n = 2$  corresponds to two-plasmon processes, etc. As the resonance with  $n = 1$  is well-known already from linear theory, we will here be concerned with  $n \geq 2$ , and assume that the condition  $k < k_{cr}$  is fulfilled, so that one-plasmon processes are forbidden.

Specifically we will focus on two-plasmon and three-plasmon processes, that can be studied if we include up to cubic terms in an amplitude expansion. The damping rates of Langmuir waves, which decrease with the decaying amplitude, are found both for  $n = 2$  and  $n = 3$ , and the evolution of the Wigner function is computed numerically.

We take as our starting point the Wigner-Moyal equation for electrons

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f - \frac{iqm^3}{\hbar} \int \frac{d^3 \mathbf{r}' d^3 \mathbf{v}'}{(2\pi\hbar)^3} e^{i\mathbf{r}' \cdot (\mathbf{v} - \mathbf{v}')m/\hbar} [\Phi(\mathbf{r} + \frac{\mathbf{r}'}{2}) - \Phi(\mathbf{r} - \frac{\mathbf{r}'}{2})] f(\mathbf{r}, \mathbf{v}', t) = 0 \quad (1)$$

Here  $f$  is the Wigner function,  $q = -|e|$  is the electron charge, and we have restricted ourselves to the electrostatic case, where  $\Phi$  is the scalar potential. The system

is closed with Poisson's equation,

$$-\nabla^2 \Phi = \frac{q}{\epsilon_0} \int d^3 \mathbf{v} f - qn_i \quad (2)$$

where  $n_i$  is a constant neutralizing ion background.

First we introduce a plane wave with a weakly time-dependent amplitude

$$\Phi = \Phi_1(t) \exp[i(kz - \omega t)] + \text{c. c.} \quad (3)$$

where c. c. denotes complex conjugate, and we take the Wigner function to be a periodic function

$$f = f_0 + \sum_{n=1}^{\infty} f_n(t) \exp[in(kz - \omega t)] + \text{c. c.} \quad (4)$$

The time dependence of the amplitudes  $\Phi_1, f_n$  is assumed to be slow compared to  $\omega$ . Here  $f_0 = F_0 + \tilde{f}_0(t)$  with  $F_0$  the background distribution, given by the Fermi-Dirac distribution at  $T = 0$ ,

$$F_0 = \begin{cases} 2m^3/(2\pi\hbar)^3 & |v| \leq v_F \\ 0 & |v| > v_F \end{cases}$$

where  $v_F$  is the Fermi velocity.

Inserting these *Ansätze* into (1) and averaging over a wave period we will obtain a hierarchy of equations for the Fourier components of the Wigner function. With a potential of the form (3) the integral over  $\mathbf{r}'$  in (1) will produce a linear combination of Dirac delta functions,

$$\int d^3\mathbf{v}' [\hat{\Phi} \exp(i(kz - \omega t)) (\delta(\mathbf{v} - \mathbf{v}' + \mathbf{v}_q) - \delta(\mathbf{v} - \mathbf{v}' - \mathbf{v}_q))] f$$

Since the problem is 1-dimensional in velocity space, we will integrate over  $v_x$  and  $v_y$  and work with the reduced Wigner function  $g(v_z)$ . The reduced background distribution is

$$G_0 = \max[2\pi m^3(v_F^2 - v_z^2)/(2\pi\hbar)^3, 0] \quad (5)$$

Matching the different frequency harmonics we find the equations

$$\partial_t g_0 = -\frac{iq}{\hbar} (\Phi \overset{\leftrightarrow}{D}_1 g_1^* - \Phi^* \overset{\leftrightarrow}{D}_1 g_1) \quad (6a)$$

$$\partial_t g_1 - i\delta\omega g_1 = -\frac{iq}{\hbar} (\Phi \overset{\leftrightarrow}{D}_1 (g_0 + G_0) - \Phi^* \overset{\leftrightarrow}{D}_1 g_2) \quad (6b)$$

$$\partial_t g_n - in\delta\omega g_n = -\frac{iq}{\hbar} (\Phi \overset{\leftrightarrow}{D}_1 g_{n-1} - \Phi^* \overset{\leftrightarrow}{D}_1 g_{n+1}) \quad (6c)$$

where  $\delta\omega = \omega - kv_z$ , (6c) is for  $n > 1$  and we have introduced the velocity shift operator

$$(\overset{\leftrightarrow}{D}_n g)(v_z) = g(v_z + nv_q) - g(v_z - nv_q) \quad (7)$$

where  $v_q = \hbar k/2m$  is a quantum velocity shift. To go to cubic order in the amplitude, we also need to include a second harmonic  $\Phi_2(t) \exp(2i(kz - \omega t)) + \text{c. c.}$  in the potential. This will give additional source terms in Eq. (6a)–(6c) of the form  $-\frac{iq}{\hbar} (\Phi_2 \overset{\leftrightarrow}{D}_2 g_{n-2} - \Phi_2^* \overset{\leftrightarrow}{D}_2 g_{n+2})$  (where  $g_{-n} = g_n^*$ ).

As we will see, the repeated action of the velocity shift operators  $\overset{\leftrightarrow}{D}_1$  and  $\overset{\leftrightarrow}{D}_2$  will induce resonances for velocities

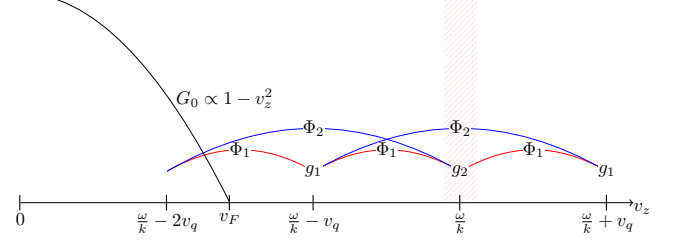


Figure 1. How the components of the Wigner function couple in the 2-plasmon damping process. The resonant region is hatched in red.

fulfilling  $\omega - kv_z - nv_q \simeq 0$  ( $n = 1, 2, 3, \dots$ ). As this condition coincides with  $\omega_a - \omega_b = n\omega$  and  $k_a - k_b = nk$ , where  $\hbar k_b/m = v_z$  is the resonant velocity and  $\omega_{a,b} = \hbar k_{a,b}^2/2m$ , in accordance with the free particle dispersion relation, we deduce that these resonances correspond to  $n$ -plasmon interactions.

Unless we are close to the resonant velocity, we can treat  $\partial_t g_1$  and  $\partial_t g_n$  as small corrections in (6b) and (6c), solving for the  $g_1$  and  $g_n$  to lowest order simply by dividing with  $-i\delta\omega$  and  $-in\delta\omega$ . We refer to the part of velocity space where this approximation is applicable as the non-resonant region, and the part where the time derivative is essential for the solution is referred to as the resonant region. We first concentrate on the two-plasmon case [14], where  $\omega/k - v_q > v_F$  (no linear resonances) but  $\omega/k - 2v_q < v_F$ . In this case only  $g_2$  couples directly to the background distribution, and it will be resonant when  $\delta\omega \simeq 0$  in the left hand side of (6c), which means that resonantly  $g_2$  couples to  $G_0(\omega/k \pm 2v_q)$  through the velocity shift operator of the second harmonic term  $\propto \Phi_2$ . Since  $g_2(v_z)$  couples to  $g_1(v_z \pm v_q)$ ,  $g_1$  will need to be treated as resonant in two regions. Figure 1 represents the two-plasmon process schematically.

The size of the resonant region is not sharply defined, but it should be taken large enough, i.e. such that  $g_2$  is negligible near the edge of the resonant region as compared to the center. We will see below that for small initial amplitudes, this is indeed possible while keeping the resonant region small, e.g., taking it to be  $[v_{\text{res}} - 0.04v_F, v_{\text{res}} + 0.04v_F]$ . Provided this is fulfilled, the numerical results are not sensitive to the exact choice of the resonant region.

The second harmonic part of Poisson's equation is

$$4k^2 \Phi_2 = \frac{q}{\epsilon_0} \int_{\text{nr}} g_2 dv_z + \frac{q}{\epsilon_0} \int_{\text{res}} g_2 dv_z. \quad (8)$$

In the non-resonant region the time derivative on  $g_2$  is negligible, and we can substitute the linear expression for  $g_1$  in the evolution equation (6c) to get

$$4k^2\Phi_2 + \frac{q^2\Phi_2}{\epsilon_0\hbar} \int_{\text{nr}} \frac{\overleftrightarrow{D}_2 G_0 dv_z}{2(\omega - kv_z)} = \frac{q^3\Phi_1^2}{2\hbar^2\epsilon_0} \sum_{\pm} \int_{\text{nr}} \frac{G_0(v_z + 2v_q) - G_0(v_z)}{(\omega - kv_z)[\omega - k(v_z \pm v_q)]} + \frac{q}{\epsilon_0} \int_{\text{res}} g_2 dv_z dv_z. \quad (9)$$

We identify the LHS of Eq. (9) as  $4k^2 D(2\omega, 2k)\Phi_2$ , where  $D(\omega, k)$  is the linear dispersion function. By a partial fractions decomposition, the coefficient for  $\Phi_2$  can be written as a sum of terms of the form  $\int_{\text{nr}} G_0(v_z)/(\omega - kv_z + mv_q)$  where  $m$  is an integer. This sum can then be related to the dispersion function. For brevity we introduce the notation  $D_n = D(n\omega, nk) = 1 + \chi_n = 1 + \chi(n\omega, nk)$  where  $\chi$  is the susceptibility. Using that  $D(\omega, k) = D_1 = 0$ , Eq. (9) can be written

$$(1 + \chi_2)\Phi_2 = \frac{q(\chi_2 - 1/4)}{2\hbar kv_q} \Phi_1^2 + \frac{q}{\epsilon_0 k^2} \int_{\text{res}} g_2 dv_z. \quad (10)$$

Now we turn to the non-linear back-reaction on  $\Phi_1$ . Poisson's equation for the first harmonic gives

$$\frac{\epsilon_0 k^2}{q} \Phi_1 = \int g_1 dv_z = \int_{\text{res}} g_1 dv_z + \int_{\text{nr}} g_1 dv_z. \quad (11)$$

In the non-resonant region we write

$$(\omega - kv_z)g_1 = -i \frac{\partial g_1}{\partial t} + \frac{q\Phi_1}{\hbar} \overleftrightarrow{D}_1 G_0 + \frac{q\Phi_2}{\hbar} \overleftrightarrow{D}_2 g_1^* - \frac{q\Phi_1^*}{\hbar} \overleftrightarrow{D}_1 g_2. \quad (12)$$

We can justify using the lowest order linear result  $g_1^L \propto \Phi_1/(\omega - kv_z)$  in the term  $-i\partial_t g_1$  by counting orders in an amplitude expansion. Since  $\Phi_2$  is a second order quantity, the non-linear corrections to  $g_1$  are third order and their time derivatives are one order higher in this expansion scheme. Then, if  $\omega, k$  satisfy the linear dispersion relation, we are led to the evolution equation for  $\Phi_1$ ,

$$ik^2 \frac{\partial D}{\partial \omega} \frac{\partial \Phi_1}{\partial t} = \frac{q}{\hbar} \Phi_1^* \Phi_2 I_{\text{nr}} + \int_{\text{res}} g_1 dv \quad (13)$$

Here the quantity  $I_{\text{nr}}$  is again a sum of terms of the form  $\int_{\text{nr}} G_0(v_z)/(\omega - kv_z + mv_q)$  where  $m$  is an integer, obtained by substituting the linear value for  $g_1$  and  $g_2$  to second order. In the resonant region we substitute for  $g_1$  according to (12) but drop the time derivative. Since only  $g_2$  is resonant, most of the integrals in  $I_{\text{nr}}$  then become integrals over all of velocity space and can be expressed in terms of the susceptibility  $\chi$ . An explicit, but lengthy, expression for  $\chi$ , where the velocity integral is solved, can be found in Ref. [10].

As for the resonant regions we note that in each part of the resonant region only one of the the terms in  $\overleftrightarrow{D} g_2$  contributes, so in terms of a variable  $\tilde{v} = v_z - \frac{\omega}{k}$  we have

$$ik^2 \frac{\partial D}{\partial \omega} \frac{\partial \Phi_1}{\partial t} = \frac{q\Phi_1^* \Phi_2 k F}{\hbar v_q} - \frac{q^2 \Phi_1^*}{m\epsilon_0} \int_{\text{res}} \frac{g_2 d\tilde{v}}{v_q^2 - \tilde{v}^2}. \quad (14)$$

where the coefficient  $F$  is

$$F = \frac{3}{2}(1 - 27\chi_3) + 16\chi_2 + O(\epsilon), \quad (15)$$

with the ordo standing for corrections due to the domains of integration variously including or excluding the resonant region. However, these corrections are small (because the resonant region is) and our conclusions are not sensitive to the precise value of  $F$ .

Next we introduce normalized dimensionless variables by  $t \mapsto t\gamma$ ,  $v_z \mapsto (v - \omega/k)/v_F$ ,  $\Phi \mapsto q\Phi/(\hbar\gamma)$  and  $g \mapsto q^2 g v_q/(\epsilon_0 \hbar k^2 \gamma)$ . As the evolution of the system occurs on a time-scale much longer than the natural scales  $(\omega^{-1}, \omega_p^{-1})$  we take an arbitrary value of  $\gamma$ , giving an arbitrary parameter  $\alpha = kv_F/\gamma$  in the equations. Then Eqs. (10) and (14) take the form

$$\Phi_2 = \frac{v_F}{v_q(1 + \chi_2)} \left( \frac{\chi_2 - 1/4}{2\alpha} \Phi_1^2 + \int g_2 dv \right) \quad (16)$$

$$i \frac{\partial D}{\partial \tilde{v}_\phi} \frac{\partial \Phi_1}{\partial t} = F \frac{v_F}{v_q} \Phi_1^* \Phi_2 - 2\Phi_1^* \int \frac{g_2 dv}{v_q^2 - v^2} \quad (17)$$

and Eq. (6c) becomes

$$\frac{\partial g_2}{\partial t} - 2i\alpha v g_2 = i\Phi_1^2 \frac{G(v - 2v_q)}{\alpha(v - v_q)} - i\Phi_2 G_0(v - 2v_q) \quad (18)$$

where the integrals are over the resonant region. The normalized background distribution is given by

$$G_0(v) = \frac{\alpha\beta}{2} \max(1 - v^2, 0) \quad (19)$$

where  $\beta = e^2 v_F/(16\pi^2 \hbar \epsilon_0 v_q^2)$ .

Next we consider three-plasmon resonances, which become dominant for slightly longer wavelengths, such that the lowest resonance  $\omega - kv_z - nv_q = 0$  occurs for  $n = 3$ , for some velocity  $v_z < v_F$ . As indicated by the previous case of two-plasmon resonances, the nonlinear frequency shift only plays a modest role for the damping rate. Thus, in order to simplify the calculation, we here neglect the nonlinear frequency shift term and focus solely on the wave damping due to the three-plasmon resonance. Over all the calculations use the same methods as the two-plasmon case but in this case the important component of  $g$  is  $g_1$  near  $\omega - kv_z = 0$ . The process is presented schematically in Figure 2.

With the same type of derivation as for the 2-plasmon damping, we find the set of normalized equations

$$i \frac{\partial D}{\partial v_\phi} \frac{\partial \Phi_1}{\partial t} = \frac{\alpha}{v_q} \int g_1 dv \quad (20)$$

$$\frac{\partial g_1}{\partial t} - i\alpha v g_1 = \frac{\chi_2 - 1/4}{1 + \chi_2} \frac{|\Phi_1|^2}{v_q^2 \alpha^2} (a\Phi_1 G_0(v) + bv g_1) \quad (21)$$

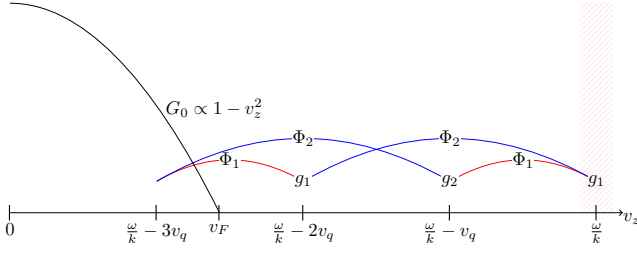


Figure 2. How the components of the Wigner function couple in the 3-plasmon damping process. The resonant region is hatched in red.

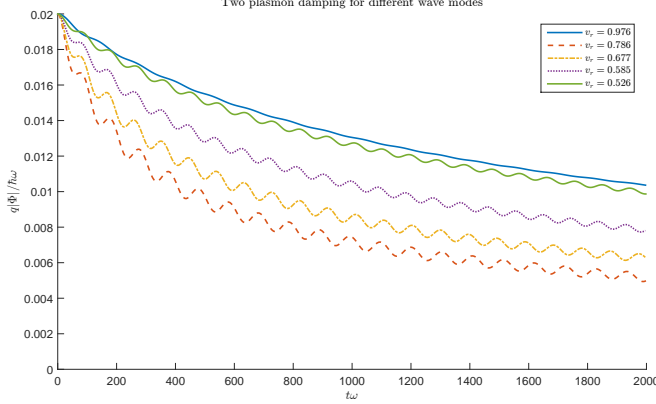


Figure 3. Two-plasmon damping for different wave modes. Each mode decays roughly as  $|\Phi(t)| = |\Phi(0)|/(1 + t/t_0)^{1/2}$ , for different values of  $t_0$ . The small scale oscillations of each curve are not a physical effect, but rather artefacts of the resonant region approximation.

where the coefficients are given by  $a = \frac{3}{2} + O(v/v_q)$  and  $b = 1 + O(v^2/v_q^2)$ . We omit the higher order terms as we have already made approximations at least this crude, and we confirm numerically that this omission shifts the shape of  $g_1$  somewhat, but does not affect the damping rate significantly.

To obtain proper values for all parameters in the two-plasmon and three-plasmon system respectively, we need to solve the linear dispersion relation  $\omega(k)$ , which serves as the input for  $\partial D/\partial v_\phi$ ,  $\chi_2$ , etc. This problem has been solved in Ref. [10] and some further comments are given in Ref. [15].

Next we study the two-plasmon system (16)–(18) and the three-plasmon system (20)–(21) numerically. We have implemented a Runge-Kutta scheme [16], fourth order in time, intended for such problems and obtained results stable against changing both the width of the resonant region and the resolution.

The damping of the wave amplitude scales the same in the two-plasmon and the three-plasmon case, and in both cases the following analytical fit

$$|\Phi(t)| = |\Phi(0)|/(1 + t/t_0)^{1/2} \quad (22)$$

is a good approximation. Here  $t_0$  is a characteristic

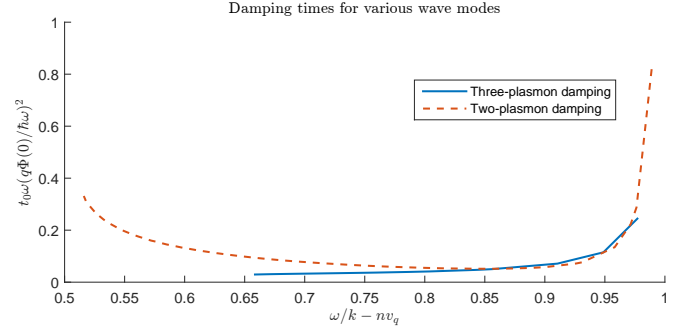


Figure 4. Damping times for various wave modes for both processes considered, as a function of the velocity of the resonant particles. The three-plasmon curve does not extend over the same region as the two-plasmon curve because of the shape of the dispersion relation. See [15] for details.

damping time that scales as

$$t_0 = C(v_q, \omega/k) \left| \frac{\hbar\omega}{q\Phi(0)} \right|^2 \frac{1}{\omega}. \quad (23)$$

The amplitude evolution for some typical cases are shown in Figure 3. The factor  $C$  varies between 0.03 and 0.5, depending on the precise location of the resonance, as shown in Figure 4. The damping time does not decrease monotonously as the resonance is moved further into the bulk of the background distribution as one would perhaps expect *prima facie*; this is explained by that the coefficients in (16)–(18) depend on  $k$ .

Fig. 3 also shows amplitude oscillations. These are due to a frequency shift that is an artefact of taking a finite resonant region. Taking a larger resonant region decreases the oscillations, but if the resonant region has to be made very large, the approximations we have used cannot be justified anymore. For the three-plasmon case, these artefacts are much smaller.

The term  $\propto \Phi_1^2$  in Eq. (16) gives a term  $\propto \Phi_1 |\Phi_1|^2$  when the expression for  $\Phi_2$  is substituted into Eq. (17), i.e., a nonlinear frequency shift term. However, this term is relatively unimportant and solving Eqs. (16)–(18) including or not including this term, does not alter the damping rate significantly. Thus the omission of the nonlinear frequency shift term when studying three-plasmon processes is justified – at least as far as we are only concerned with the evolution of the wave amplitude. If we shift our interest to the evolution of the resonant particles, see Fig. 5, the situation is somewhat changed. Here a frequency shift changes the exact location of the resonance. Since the approximation of a finite resonant region also gives rise to a frequency shift, this is still seen in Fig. 5. This effect is much less pronounced in the three-plasmon case, as the size of the resonant region is less relevant in this case.

In the present letter we have studied wave-particle interaction due to multi-plasmon resonances. While multi-

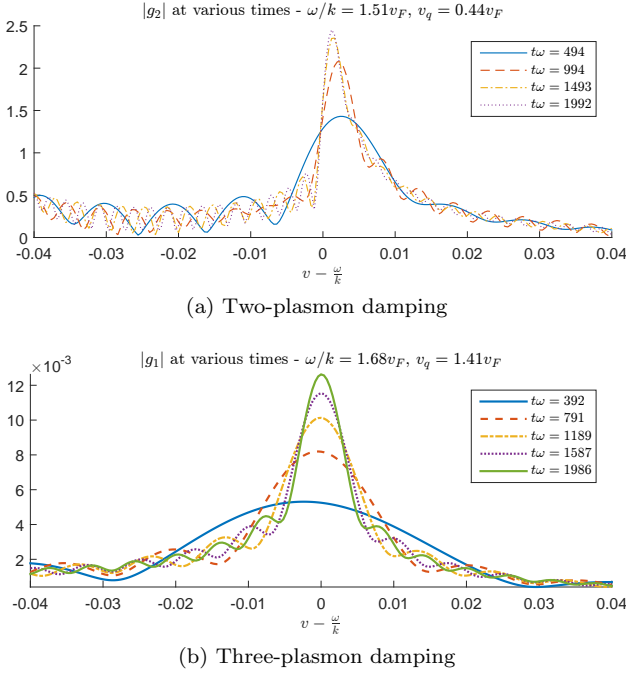


Figure 5. The absolute value of the resonant Fourier component of the Wigner function at various times. It is seen that the Wigner function is significant mainly near the resonance. At late times fine structures develop in velocity space.

plasmon resonances can be of significance also for hot plasmas (i.e.  $T_F \gtrsim T$ ) if the plasma density is high enough, these effects are of particular importance for low-temperature plasmas ( $T \ll T_F$ ). The contribution from a finite, but low, temperature is mainly in how the locations of resonances are affected by the linear dispersion relation [13]. For  $T \ll T_F$ , two plasmon processes occur for wavenumbers  $k_2 < k < k_{cr}$ , where  $k_2 \simeq 0.69\omega_p/v_F$  and  $k_{cr} = 0.98\omega_p/v_F$ . Whenever two-plasmon processes occur, we also have three-plasmon processes, which give a damping of comparable magnitude. A more precise comparison of two- and three-plasmon damping is made in Fig. 4. For  $k_3 < k < k_2$ , where  $k_3 = 0.60\omega_p/v_F$ , three-plasmon damping dominates and two-plasmon processes are absent. This holds because  $n$ -plasmon processes with  $n \geq 4$  are of higher order in an amplitude expansion. For multi-plasmon processes with  $n = 2$  or  $n = 3$ , the damping decreases with the amplitude, as given by Eq. (23). Thus the present processes are of limited importance for very small amplitudes, in which case collisional damping will be the dominant damping mechanism when  $k < k_{cr}$ . Nevertheless, the processes studied here can be important for broad classes of systems. In particular multi-plasmon resonances can play a role in dense plasmas such as e.g. metallic plasmas, white dwarf stars and inertially confined plasmas. More generally, the methods used here

can be used to study resonant interaction involving multiple wave quanta of other types of waves, which can be of significance for wave-particle damping of any wave mode, provided the wavelength is not much longer than the de Broglie wavelength. For example, resonant wave-particle interaction in high-density plasmas involving intense short-wavelength electromagnetic radiation can be possible for  $n \geq 1$ , potentially leading to new opportunities for particle acceleration. However, this remains a project for further study.

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  - [14] As we will see, when two-plasmon are allowed three-plasmon will also take place. Moreover, the damping rate due to these processes are of the same magnitude. Thus in principle two- and three-plasmon processes should be studied simultaneously. However, the physics is easier to understand in case the processes are studied in isolation. In practice the wave damping due to the two processes can be added together afterwards, which is the approach we will chose. Also note that three-plasmon processes always can be studied in isolation, since there is a regime that forbids two-plasmon processes but allows three-plasmon processes.
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